

Convergence of the Gauss-Newton method for a special class of systems of equations under a majorant condition

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Abstract

In this paper, we study the Gauss-Newton method for a special class of systems of nonlinear equation. Under the hypothesis that the derivative of the function under consideration satisfies a majorant condition, semi-local convergence analysis is presented. In this analysis the conditions and proof of convergence are simplified by using a simple majorant condition to define regions where the Gauss-Newton sequence is “well behaved”. Moreover, special cases of the general theory are presented as applications.

Keywords: Gauss-Newton method; majorant condition; nonlinear systems of equations; semi-local convergence.

1 Introduction

Consider the *systems of nonlinear equations*

$$F(x) = 0, \tag{1}$$

where $F : \Omega \rightarrow \mathbb{R}^m$ is a continuously differentiable function and $\Omega \subseteq \mathbb{R}^n$ is an open set.

When $F'(x)$ is invertible, the Newton method and its variant (see [5, 6, 7, 10]) are the most efficient methods known for solving (1). However, when $F'(x)$ is not necessarily invertible, a generalized Newton method, called the Gauss-Newton method (see [4, 8, 9]), defined by

$$x_{k+1} = x_k - F'(x_k)^\dagger F(x_k), \quad k = 0, 1, \dots,$$

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where $F'(x_k)^\dagger$ denotes the Moore-Penrose inverse of the linear operator $F'(x_k)$, finds least squares solutions of (1) which may or may not be solutions of (1). These least squares solutions are related to the nonlinear least squares problem

$$\min_{x \in \Omega} \|F(x)\|^2, \quad (2)$$

that is, they are stationary points of $G(x) = \|F(x)\|^2$. It is worth noting that, if $F'(x)$ is surjective, then least squares solutions of systems of nonlinear equations are also solutions of systems of nonlinear equations.

We shall consider the same special class of systems of nonlinear equations studied in [11, 12, 14], i.e., systems of nonlinear equations where the function F under consideration satisfies

$$\|F'(y)^\dagger(I_{\mathbb{R}^m} - F'(x)F'(x)^\dagger)F(x)\| \leq \kappa\|x - y\|, \quad \forall x, y \in \Omega \quad (3)$$

for some $0 \leq \kappa < 1$ and $I_{\mathbb{R}^m}$ denotes the identity operator on \mathbb{R}^m . This special class of nonlinear systems of equation contains underdetermined systems with surjective derivatives, because when $F'(x)$ is surjective we can prove that $\kappa = 0$ in (3).

In recent years, papers have addressed the issue of convergence of the Newton method, including the Gauss-Newton method, by relaxing the assumption of Lipschitz continuity of the derivative (see [2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 17, 18] and references therein). These new assumptions also allow us to unify previously unrelated convergence results, namely results for analytical functions (α -theory or γ -theory) and the classical results for functions with Lipschitz derivative. The main new conditions that relax the condition of Lipschitz continuity of the derivative include the majorant condition, which we will use, and Wang's condition, introduced in [16] and used for example in [14, 17, 18] to study the Gauss-Newton method. In fact, under the hypothesis in this paper, it can be shown that these conditions are equivalent. However, the formulation as a majorant condition is in a sense better than Wang's condition, as it provides a clear relationship between the majorant function and the nonlinear function under consideration. Besides, the majorant condition provides a simpler proof of convergence.

Following the ideas of the semi-local convergence analysis in [8, 10], we will present a new semi-local convergence analysis of the Gauss-Newton method for solving (1), where F satisfies (3), under a majorant condition. The convergence analysis presented here communicates the conditions and proof in a quite simple manner. This is possible thanks to our majorant condition and to a demonstration technique introduced in [10] which, instead of looking only to the sequence generated, identifies regions where, for the problem under consideration, the Gauss-Newton sequence is well behaved, as compared with a method applied to an auxiliary function associated with the majorant function. Moreover, two unrelated previous results relating to the Gauss-Newton method are unified, namely, results for analytical functions

under an α -condition and the classical result for functions with Lipschitz derivative. Besides, convergence results for underdetermined systems with surjective derivatives will be also given.

The paper is organized as follows. Sect. 1.1 lists some notations and basic results used in the presentation. Sect. 2 states and proves the main results. Finally, special cases of the general theory are presented as applications in Sect. 3.

1.1 Notation and auxiliary results

The following notations and results are used throughout this presentation. Let \mathbb{R}^n be with a norm $\|\cdot\|$. The open and closed balls at $a \in \mathbb{R}^n$ and radius $\delta > 0$ are denoted, respectively by

$$B(a, \delta) := \{x \in \mathbb{R}^n; \|x - a\| < \delta\}, \quad B[a, \delta] := \{x \in \mathbb{R}^n; \|x - a\| \leq \delta\}.$$

Given a linear operator $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ (or an $n \times m$ matrix), the Moore-Penrose inverse of A is the linear operator $A^\dagger : \mathbb{R}^m \rightarrow \mathbb{R}^n$ (or an $m \times n$ matrix) which satisfies:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A,$$

where A^* denotes the adjoint of A . The Kernel and image of A are denoted by $\text{Ker}(A)$ and $\text{im}(A)$, respectively. It is easily seen from the definition of the Moore-Penrose inverse that

$$AA^\dagger = \Pi_{\text{Ker}(A)^\perp}, \quad A^\dagger A = \Pi_{\text{im}(A)}, \quad (4)$$

where Π_E denotes the projection of \mathbb{R}^n onto subspace E .

We use $I_{\mathbb{R}^m}$ to denote the identity operator on \mathbb{R}^m . If A is surjective, then

$$A^\dagger = A^*(AA^*)^{-1}, \quad AA^\dagger = I_{\mathbb{R}^m}, \quad (AA^\dagger)^\dagger = AA^\dagger. \quad (5)$$

Lemma 1. (*Banach's Lemma*) Let $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous linear operator. If $\|B - I_{\mathbb{R}^n}\| < 1$, then B is invertible and $\|B^{-1}\| \leq 1/(1 - \|B - I_{\mathbb{R}^n}\|)$.

Proof. See the proof of Lemma 1, p.189 of Smale [15] with $A = I_{\mathbb{R}^n}$ and $c = \|B - I_x\|$. \square

The next lemma is proved on p.43 of [13] (see also [1]). It is on the perturbation of the Moore-Penrose inverse of A .

Lemma 2. Let $A, B : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be continuous linear operators. Assume that

$$1 \leq \text{rank}(B) \leq \text{rank}(A), \quad \|A^\dagger\| \|A - B\| < 1.$$

Then

$$\text{rank}(A) = \text{rank}(B), \quad \|A^\dagger\| \leq \frac{\|B^\dagger\|}{1 - \|B^\dagger\| \|A - B\|}.$$

2 Semi-local analysis for the Gauss-Newton method

Our goal is to state and prove a semi-local theorem of the Gauss-Newton method for solving nonlinear systems of equations, where the function under consideration satisfies (3). First, we will prove that this theorem holds for an auxiliary function associated with the majorant function. Then, we will prove well-definedness of the Gauss-Newton method and convergence. Convergence rates will also be established. The statement of the theorem is:

Theorem 3. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $F : \Omega \rightarrow \mathbb{R}^m$ a continuously differentiable function. Suppose that*

$$\|F'(y)^\dagger(I_{\mathbb{R}^m} - F'(x)F'(x)^\dagger)F(x)\| \leq \kappa\|x - y\|, \quad \forall x, y \in \Omega \quad (6)$$

for some $0 \leq \kappa < 1$. Take $x_0 \in \Omega$ such that $\beta := \|F'(x_0)^\dagger F(x_0)\| > 0$, $F'(x_0) \neq 0$ and

$$\text{rank}(F'(x)) \leq \text{rank}(F'(x_0)), \quad \forall x \in \Omega. \quad (7)$$

Suppose that there exist $R > 0$ and a continuously differentiable function $f : [0, R) \rightarrow \mathbb{R}$ such that, $B(x_0, R) \subseteq \Omega$,

$$\|F'(x_0)^\dagger\| \|F'(y) - F'(x)\| \leq f'(\|y - x\| + \|x - x_0\|) - f'(\|x - x_0\|), \quad (8)$$

for any $x, y \in B(x_0, R)$, $\|x - x_0\| + \|y - x\| < R$,

h1) $f(0) = 0$, $f'(0) = -1$;

h2) f' is convex and strictly increasing.

Take $\lambda \geq 0$ such that $\lambda \geq -\kappa f'(\beta)$ and consider the auxiliary function $h_{\beta, \lambda} : [0, R) \rightarrow \mathbb{R}$,

$$h_{\beta, \lambda}(t) := \beta + \lambda t + f(t). \quad (9)$$

If $h_{\beta, \lambda}$ satisfies

h3) $h_{\beta, \lambda}(t) = 0$ for some $t \in (0, R)$,

then $h_{\beta, \lambda}(t)$ has a smallest zero $t_* \in (0, R)$, the sequences for solving $h_{\beta, \lambda}(t) = 0$ and $F(x) = 0$, with starting point $t_0 = 0$ and x_0 , respectively,

$$t_{k+1} = t_k - h'_{\beta, 0}(t_k)^{-1} h_{\beta, \lambda}(t_k), \quad x_{k+1} = x_k - F'(x_k)^\dagger F(x_k), \quad k = 0, 1, \dots, \quad (10)$$

are well defined, $\{t_k\}$ is strictly increasing, is contained in $[0, t_*)$, and converges to t_* , $\{x_k\}$ is contained in $B(x_0, t_*)$, converges to a point $x_* \in B[x_0, t_*]$ such that $F'(x_*)^\dagger F(x_*) = 0$ and

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \quad \|x_* - x_k\| \leq t_* - t_k, \quad k = 0, 1, \dots, \quad (11)$$

$$\|x_{k+1} - x_k\| \leq \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2} \|x_k - x_{k-1}\|^2, \quad k = 1, 2, \dots. \quad (12)$$

Moreover, if $\lambda = 0$ ($\lambda = 0$ and $h'_{\beta,0}(t_*) < 0$), the sequences $\{t_k\}$ and $\{x_k\}$ converge Q -linearly and R -linearly (Q -quadratically and R -quadratically) to t_* and x_* , respectively.

Remark 1. It is easily seen that the best choice of λ is the smallest possible. Hence, if $f'(\beta) \leq 0$ then $\lambda = -\kappa f'(\beta)$ is the best choice. Moreover, since $-f'(\beta) < -f'(0) = 1$ (**h2**), a possible choice for λ is κ , despite not being the best.

Remark 2. If $F'(x)$ is surjective, it follows from the second equation in (5) that $F'(x)F'(x)^\dagger = I_{\mathbb{R}^m}$. Thus, we can take $\lambda = 0$, because F satisfies (6) with $\kappa = 0$. Therefore, in this case, Theorem 3 extends the results obtained by Ferreira and Svaiter in Theorem 2 of [10].

From now on, we assume that the hypotheses of Theorem 3 hold.

2.1 The auxiliary function and sequence $\{t_k\}$

In this section, we will study the auxiliary function, $h_{\beta,\lambda}$, which is associated with the majorant function, f , and prove all results regarding only the sequence $\{t_k\}$. Remember that a function that satisfies (8), **h1** and **h2** is called a majorant function for the function F on $B(x_0, R)$. More details about the majorant condition can be found in [4, 5, 6, 7, 8, 9, 10].

Proposition 4. *The following statements hold:*

- i) $h_{\beta,\lambda}(0) = \beta > 0$, $h'_{\beta,\lambda}(0) = \lambda - 1$;
- ii) $h'_{\beta,\lambda}$ is convex and strictly increasing.

Proof. It follows from the definition in (9) and assumptions **h1** and **h2**. □

Proposition 5. *The function $h_{\beta,\lambda}$ has a smallest root $t_* \in (0, R)$, is strictly convex, and*

$$h_{\beta,\lambda}(t) > 0, \quad h'_{\beta,0}(t) < 0, \quad t < t - h_{\beta,\lambda}(t)/h'_{\beta,0}(t) < t_*, \quad \forall t \in [0, t_*]. \quad (13)$$

Moreover, $h'_{\beta,0}(t_*) \leq 0$.

Proof. As $h_{\beta,\lambda}$ is continuous in $[0, R)$ and have a zero there (**h3**), it must have a smallest zero t_* , which is greater than 0 because $h_{\beta,\lambda}(0) = \beta > 0$. Since $h'_{\beta,\lambda}$ is strictly increasing by item **ii** of Proposition 4, $h_{\beta,\lambda}$ is strictly convex.

The first inequality in (13) follows from the assumption $h_{\beta,\lambda}(0) = \beta > 0$ and the definition of t_* as the smallest root of $h_{\beta,\lambda}$. Since $h_{\beta,\lambda}$ is strictly convex,

$$0 = h_{\beta,\lambda}(t_*) > h_{\beta,\lambda}(t) + h'_{\beta,\lambda}(t)(t_* - t), \quad t \in [0, R), \quad t \neq t_*. \quad (14)$$

If $t \in [0, t_*)$ then $h_{\beta,\lambda}(t) > 0$ and $t_* - t > 0$, which, combined with (14) yields $h'_{\beta,\lambda}(t) < 0$ for all $t \in [0, t_*)$. Hence, using $\lambda \geq 0$ and $h'_{\beta,\lambda}(t) = \lambda + h'_{\beta,0}(t)$ for all $t \in [0, t_*)$ the second inequality in (13) follows. The third inequality in (13) follows from the first and the second inequalities.

To prove the last inequality in (13), note that the division of the inequality on (14) by $-h'_{\beta,\lambda}(t)$ (which is strictly positive), together with some simple algebraic manipulations, gives

$$t - h_{\beta,\lambda}(t)/h'_{\beta,\lambda}(t) < t_*, \quad \forall t \in [0, t_*),$$

which, using the first inequality in (13) and $0 < -h'_{\beta,\lambda}(t) \leq -h'_{\beta,0}(t)$ for all $t \in [0, t_*)$, yields the desired inequality.

Since $h_{\beta,\lambda} > 0$ in $[0, t_*)$ and $h_{\beta,\lambda}(t_*) = 0$, we must have $h'_{\beta,\lambda}(t_*) \leq 0$. Thus, the last inequality of the proposition follows from the fact that $h'_{\beta,\lambda}(t_*) = \lambda + h'_{\beta,0}(t_*)$. \square

In view of the second inequality in (13), the following iteration map for $h_{\beta,\lambda}$ is well defined in $[0, t_*)$. Denoting this by $n_{h_{\beta,\lambda}}$:

$$\begin{aligned} n_{h_{\beta,\lambda}} : [0, t_*) &\rightarrow \mathbb{R} \\ t &\mapsto t - h_{\beta,\lambda}(t)/h'_{\beta,0}(t). \end{aligned} \quad (15)$$

Note that in the case where $\lambda = 0$, the sequence $n_{h_{\beta,\lambda}}$ reduces to a Newton sequence, which Ferreira and Svaiter used in [10] to obtain a semi-local convergence analysis of the Newton method under a majorant condition.

Proposition 6. *For each $t \in [0, t^*)$ it holds that $\beta \leq n_{h_{\beta,\lambda}}(t) < t_*$.*

Proof. Proposition 5 implies that $h_{\beta,\lambda}$ is convex. Hence, using item **i** of Proposition 4 it is easy to see, by using convexity properties, that $(1 - \lambda)t - \beta \geq -h_{\beta,\lambda}(t)$, which combined with $\lambda \geq 0$ gives $t - \beta \geq -h_{\beta,\lambda}(t)$. Accordingly, the above definition implies that

$$n_{h_{\beta,\lambda}}(t) - \beta = t - \frac{h_{\beta,\lambda}(t)}{h'_{\beta,0}(t)} - \beta \geq -h_{\beta,\lambda}(t) - \frac{h_{\beta,\lambda}(t)}{h'_{\beta,0}(t)} = \frac{h_{\beta,\lambda}(t)}{-h'_{\beta,0}(t)}[h'_{\beta,0}(t) + 1], \quad \forall t \in [0, t_*).$$

Proposition 4 implies that $h'_{\beta,0}(0) = -1$ and $h'_{\beta,0}$ is strictly increasing. Thus, we obtain $h'_{\beta,0}(t) + 1 \geq 0$, for all $t \in [0, t_*)$. Therefore, combining the above inequality with the first two inequalities in Proposition 5, the first inequality of proposition follows. To prove the last inequality of proposition, combine (15) with the last inequality in (13). \square

Proposition 7. *Iteration map $n_{h_{\beta,\lambda}}$ maps $[0, t^*)$ in $[0, t^*)$, and it holds that*

$$t < n_{h_{\beta,\lambda}}(t), \quad \forall t \in [0, t_*).$$

Moreover, if $\lambda = 0$ or $\lambda = 0$ and $h'_{\beta,0}(t_*) < 0$, we have the follows inequalities, respectively,

$$t_* - n_{h_{\beta,\lambda}}(t) \leq \frac{1}{2}(t_* - t), \quad t_* - n_{h_{\beta,\lambda}}(t) \leq \frac{D^- h'_{\beta,0}(t_*)}{-2h'_{\beta,0}(t_*)}(t_* - t)^2, \quad \forall t \in [0, t_*).$$

Proof. The first two statements of the proposition follow trivially for the last inequalities in (13) and (15). Now, if $\lambda = 0$, then the sequence in (15) reduces to a Newton sequence. Hence, the second part of the proof follows the same pattern as the proof of Proposition 4 of [10] with $h_{\beta,0} = f$. \square

The definition of $\{t_k\}$ in Theorem 3 is equivalent to the following one

$$t_0 = 0, \quad t_{k+1} = n_{h_{\beta,\lambda}}(t_k), \quad k = 0, 1, \dots \quad (16)$$

Therefore, using also Proposition 7 it is easy to prove that

Corollary 8. *The sequence $\{t_k\}$ is well defined, is strictly increasing, is contained in $[0, t_*)$, and converges to t_* .*

Moreover, if $\lambda = 0$ or $\lambda = 0$ and $h'_{\beta,0}(t_*) < 0$, the sequence $\{t_k\}$ converges Q -linearly or Q -quadratically to t_* , respectively, as follows

$$t_* - t_{k+1} \leq \frac{1}{2}(t_* - t_k), \quad t_* - t_{k+1} \leq \frac{D^- h'_{\beta,0}(t_*)}{-2h'_{\beta,0}(t_*)}(t_* - t_k)^2, \quad k = 0, 1, \dots$$

Hence, all statements involving only $\{t_k\}$ on Theorem 3 are valid

2.2 Convergence

In this section we will prove well definedness and convergence of the sequence $\{x_k\}$ specified on (10) in Theorem 3.

We start with two lemma that highlight the relationships between the majorant function f and the non-linear function F .

Proposition 9. *If $\|x - x_0\| \leq t < t_*$, then $\text{rank}(F'(x)) = \text{rank}(F'(x_0)) \geq 1$ and*

$$\|F'(x)^\dagger\| \leq -\|F'(x_0)^\dagger\|/h'_{\beta,0}(t).$$

In particular, $\text{rank}(F'(x)) = \text{rank}(F'(x_0))$ in $B(x_0, t_)$.*

Proof. Take $x \in B[x_0, t]$, $0 \leq t < t_*$. Using the assumptions (8), **h1**, **h2**, $f'(t) = h'_{\beta,0}(t)$ and the second inequality in (13) we obtain

$$\|F'(x_0)^\dagger\| \|F'(x) - F'(x_0)\| \leq f'(\|x - x_0\|) - f'(0) \leq f'(t) + 1 = h'_{\beta,0}(t) + 1 < 1.$$

Combining the last inequality with (7) and Lemma 2, we conclude that $\text{rank}(F'(x)) = \text{rank}(F'(x_0)) \geq 1$ and

$$\|F'(x)^\dagger\| \leq \frac{\|F'(x_0)^\dagger\|}{1 - (f'(t) + 1)} = \frac{\|F'(x_0)^\dagger\|}{-f'(t)} = -\frac{\|F'(x_0)^\dagger\|}{h'_{\beta,0}(t)}.$$

□

It is convenient to study the linearization error of F at point in Ω . For that purpose we define

$$E_F(x, y) := F(y) - [F(x) + F'(x)(y - x)], \quad y, x \in \Omega. \quad (17)$$

We will bound this error by the error in the linearization on the majorant function f

$$e_f(t, u) := f(u) - [f(t) + f'(t)(u - t)], \quad t, u \in [0, R]. \quad (18)$$

Lemma 10. *Take*

$$x, y \in B(x_0, R) \quad \text{and} \quad 0 \leq t < v < R.$$

If $\|x - x_0\| \leq t$ and $\|y - x\| \leq v - t$, then

$$\|F'(x_0)^\dagger\| \|E_F(x, y)\| \leq e_f(t, v) \frac{\|y - x\|^2}{(v - t)^2}.$$

Proof. The proof follows the same pattern as the proof of Lemma 7 of [10]. □

Proposition 9 guarantees, in particular, that $\text{rank}(F'(x)) \geq 1$ for all $x \in B(x_0, t_*)$ and, consequently, the Gauss-Newton iteration map is well-defined. Let us call G_F , the Gauss-Newton iteration map for F in that region:

$$\begin{aligned} G_F : B(x_0, t_*) &\rightarrow \mathbb{R}^n \\ x &\mapsto x - F'(x)^\dagger F(x). \end{aligned} \quad (19)$$

One can apply a *single* Gauss-Newton iteration on any $x \in B(x_0, t_*)$ to obtain $G_F(x)$ which may not belong to $B(x_0, t_*)$, or even may not belong to the domain of F . Therefore, this is enough to guarantee well definedness of only one iteration. To ensure that Gauss-Newton iterations may be repeated indefinitely, we need the following result.

First, we define some subsets of $B(x_0, t_*)$ in which, as we shall prove, the desired inclusion holds for all points in these subsets.

$$K(t) := \left\{ x \in \Omega : \|x - x_0\| \leq t, \|F'(x)^\dagger F(x)\| \leq -\frac{h_{\beta,\lambda}(t)}{h'_{\beta,0}(t)} \right\}, \quad t \in [0, t_*), \quad (20)$$

$$K := \bigcup_{t \in [0, t_*)} K(t). \quad (21)$$

In (20), $0 \leq t < t_*$, therefore, $h'_{\beta,0}(t) \neq 0$ and $\text{rank}(F'(x)) \geq 1$ in $B[x_0, t] \subset B[x_0, t_*)$ (Proposition 9). Hence, the definitions are consistent.

Lemma 11. *For each $t \in [0, t_*)$, it holds that:*

- i) $K(t) \subset B(x_0, t_*)$;
- ii) $\|G_F(G_F(x)) - G_F(x)\| \leq -\frac{h_{\beta,\lambda}(n_{h_{\beta,\lambda}}(t))}{h'_{\beta,0}(n_{h_{\beta,\lambda}}(t))} \left(\frac{\|G_F(x) - x\|}{n_{h_{\beta,\lambda}}(t) - t} \right)^2, \quad \forall x \in K(t),$
- iii) $G_F(K(t)) \subset K(n_{h_{\beta,\lambda}}(t)).$

As a consequence, $K \subset B(x_0, t_*)$ and $G_F(K) \subset K$.

Proof. Item **i** follows trivially from the definition of $K(t)$.

Take $t \in [0, t_*)$, $x \in K(t)$. Using definition (20) and the first two statements in Proposition 7 we have

$$\|x - x_0\| \leq t, \quad \|F'(x)^\dagger F(x)\| \leq -h_{\beta,\lambda}(t)/h'_{\beta,0}(t), \quad t < n_{h_{\beta,\lambda}}(t) < t_*. \quad (22)$$

Therefore

$$\begin{aligned} \|G_F(x) - x_0\| &\leq \|x - x_0\| + \|G_F(x) - x\| = \|x - x_0\| + \|F'(x)^\dagger F(x)\| \\ &\leq t - h_{\beta,\lambda}(t)/h'_{\beta,0}(t) = n_{h_{\beta,\lambda}}(t) < t_*, \end{aligned}$$

and

$$G_F(x) \in B[x_0, n_{h_{\beta,\lambda}}(t)] \subset B(x_0, t_*). \quad (23)$$

Since $G_F(x)$, $n_{h_{\beta,\lambda}}(t)$ belong to the domains of F and f , respectively, using the definitions in (15) and (19), $h_{\beta,\lambda}(t) = \beta + \lambda t + f(t)$, linearization errors (17) and (18) and some algebraic manipulation, we obtain

$$\begin{aligned} h_{\beta,\lambda}(n_{h_{\beta,\lambda}}(t)) &= h_{\beta,\lambda}(n_{h_{\beta,\lambda}}(t)) - [h_{\beta,\lambda}(t) + h'_{\beta,0}(t)(n_{h_{\beta,\lambda}}(t) - t)] \\ &= e_f(t, n_{h_{\beta,\lambda}}(t)) - \lambda h_{\beta,\lambda}(t)/h'_{\beta,0}(t) \end{aligned} \quad (24)$$

and

$$\begin{aligned} F(G_F(x)) &= F(G_F(x)) - [F(x) + F'(x)(G_F(x) - x)] + (I_{\mathbb{R}^m} - F'(x)F'(x)^\dagger)F(x) \\ &= E_F(x, G_F(x)) + (I_{\mathbb{R}^m} - F'(x)F'(x)^\dagger)F(x). \end{aligned}$$

The last equation, together with simple algebraic manipulations, implies that

$$\begin{aligned} \|F'(G_F(x))^\dagger F(G_F(x))\| &\leq \|F'(G_F(x))^\dagger\| \|E_F(x, G_F(x))\| \\ &\quad + \|F'(G_F(x))^\dagger(I_{\mathbb{R}^m} - F'(x)F'(x)^\dagger)F(x)\|. \end{aligned}$$

As $\|G_F(x) - x_0\| \leq n_{h_{\beta,\lambda}}(t)$, it follows from Proposition 9 that $\text{rank}(F'(G_F(x))) \geq 1$ and

$$\|F'(G_F(x))^\dagger\| \leq -\|F'(x_0)^\dagger\|/h'_{\beta,0}(n_{h_{\beta,\lambda}}(t)).$$

From the two latter equations and (6) we have

$$\|F'(G_F(x))^\dagger F(G_F(x))\| \leq -\frac{\|F'(x_0)^\dagger\|}{h'_{\beta,0}(n_{h_{\beta,\lambda}}(t))} \|E(x, G_F(x))\| + \kappa \|G_F(x) - x\|.$$

On the other hand, using (22), Lemma 10 and (24) we have

$$\begin{aligned} \|F'(x_0)^\dagger\| \|E_F(x, G_F(x))\| &\leq e_f(t, n_{h_{\beta,\lambda}}(t)) \left(\frac{\|G_F(x) - x\|}{n_{h_{\beta,\lambda}}(t) - t} \right)^2 \\ &\leq h_{\beta,\lambda}(n_{h_{\beta,\lambda}}(t)) \left(\frac{\|G_F(x) - x\|}{n_{h_{\beta,\lambda}}(t) - t} \right)^2 + \lambda h_{\beta,\lambda}(t)/h'_{\beta,0}(t). \end{aligned}$$

Thus, the last two equations, together with the second equation in (22), imply

$$\begin{aligned} \|F'(G_F(x))^\dagger F(G_F(x))\| &\leq \frac{-h_{\beta,\lambda}(n_{h_{\beta,\lambda}}(t))}{h'_{\beta,0}(n_{h_{\beta,\lambda}}(t))} \left(\frac{\|G_F(x) - x\|}{n_{h_{\beta,\lambda}}(t) - t} \right)^2 \\ &\quad + (\kappa + \lambda h'_{\beta,0}(n_{h_{\beta,\lambda}}(t))^{-1})(-h_{\beta,\lambda}(t)/h'_{\beta,0}(t)). \end{aligned}$$

Taking $\lambda \geq -\kappa f'(\beta)$, the second inequality in (13) and (22), we obtain

$$(\kappa + \lambda(h'_{\beta,0}(n_{h_{\beta,\lambda}}(t)))^{-1}) \leq \kappa(1 - f'(\beta)(h'_{\beta,0}(n_{h_{\beta,\lambda}}(t)))^{-1}).$$

As $f'(t) = h'_{\beta,0}(t)$, using Proposition 6, **h2** and the second inequality in (13), we have

$$\kappa(1 - f'(\beta)(h'_{\beta,0}(n_{h_{\beta,\lambda}}(t)))^{-1}) = \kappa(h'_{\beta,0}(\beta) - h'_{\beta,0}(n_{h_{\beta,\lambda}}(t)))(-h'_{\beta,0}(n_{h_{\beta,\lambda}}(t)))^{-1} \leq 0.$$

Combining the three above inequalities we conclude

$$\|F'(G_F(x))^\dagger F(G_F(x))\| \leq \frac{-h_{\beta,\lambda}(n_{h_{\beta,\lambda}}(t))}{h'_{\beta,0}(n_{h_{\beta,\lambda}}(t))} \left(\frac{\|G_F(x) - x\|}{n_{h_{\beta,\lambda}}(t) - t} \right)^2.$$

Therefore, item **ii** follows from the last inequality and (19). Now, the last inequality combined with (15), (19) and the second inequality in (22) becomes

$$\|F'(G_F(x))^\dagger F(G_F(x))\| \leq \frac{-h_{\beta,\lambda}(n_{h_{\beta,\lambda}}(t))}{h'_{\beta,0}(n_{h_{\beta,\lambda}}(t))}.$$

This result, together with (23), shows that $G_F(x) \in K(n_{h_{\beta,\lambda}}(t))$, which proves item **iii**.

The next inclusion (first on the second part), follows trivially from definitions (20) and (21). To check the last inclusion, take $x \in K$. Then $x \in K(t)$ for some $t \in [0, t_*)$. Using item **iii** of the lemma, we conclude that $G_F(x) \in K(n_{h_{\beta,\lambda}}(t))$. To end the proof, note that $n_{h_{\beta,\lambda}}(t) \in [0, t_*)$ and use the definition of K . \square

Finally, we are ready to prove the main result of this section, which is an immediate consequence of the latter result. First note that the sequence $\{x_k\}$ (see (10)) satisfies

$$x_{k+1} = G_F(x_k), \quad k = 0, 1, \dots, \quad (25)$$

which is indeed an equivalent definition of this sequence.

Corollary 12. *The sequence $\{x_k\}$ is well defined, is contained in $B(x_0, t_*)$, converges to a point $x_* \in B[x_0, t_*]$ such that $F'(x_*)^\dagger F(x_*) = 0$, and $\{x_k\}$ and $\{t_k\}$ satisfy (11) and (12).*

Moreover, if $\lambda = 0$ ($\lambda = 0$ and $h'_{\beta,0}(t_) < 0$), the sequences $\{t_k\}$ and $\{x_k\}$ converge Q -linearly and R -linearly (Q -quadratically and R -quadratically) to t_* and x_* , respectively.*

Proof. Since $\|F'(x_0)^\dagger F(x_0)\| = \beta$, using the item **i** of the Proposition 4, we have

$$x_0 \in K(0) \subset K,$$

where the second inclusion follows trivially from (21). Using the above equation, the inclusions $G_F(K) \subset K$ (Lemma 11) and (25), we conclude that the sequence $\{x_k\}$ is well defined and lies in K . From the first inclusion in the second part of Lemma 11, we have trivially that $\{x_k\}$ is contained in $B(x_0, t_*)$.

We will prove, by induction, that

$$x_k \in K(t_k), \quad k = 0, 1, \dots \quad (26)$$

The above inclusion, for $k = 0$, is the first result in this proof. Assume now that $x_k \in K(t_k)$. Thus, using item **iii** of Lemma 11, (16) and (25), we conclude that $x_{k+1} \in K(t_{k+1})$, which completes the induction proof of (26).

Now, using (26) and (20), we have

$$\|F'(x_k)^\dagger F(x_k)\| \leq -h_{\beta, \lambda}(t_k)/h'_{\beta, 0}(t_k), \quad k = 0, 1, \dots,$$

which, using (10), becomes

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \quad k = 0, 1, \dots \quad (27)$$

So, the first inequality in (11) holds. As $\{t_k\}$ converges to t_* , the last inequality implies that

$$\sum_{k=k_0}^{\infty} \|x_{k+1} - x_k\| \leq \sum_{k=k_0}^{\infty} t_{k+1} - t_k = t_* - t_{k_0} < +\infty,$$

for any $k_0 \in \mathbb{N}$. Hence, $\{x_k\}$ is a Cauchy sequence in $B(x_0, t_*)$, and so converges to some $x_* \in B[x_0, t_*]$. The last inequality also implies that the second inequality in (11) holds.

To prove that $F'(x_*)^\dagger F(x_*) = 0$, note that, with simple algebraic manipulation, (6) and (10), we obtain

$$\begin{aligned} \|F'(x_*)^\dagger F(x_k)\| &\leq \|F'(x_*)^\dagger (I - F'(x_k)F'(x_k)^\dagger)F(x_k)\| \\ &\quad + \|F'(x_*)^\dagger\| \|F'(x_k)F'(x_k)^\dagger F(x_k)\| \\ &\leq \kappa \|x_k - x_*\| + \|F'(x_*)^\dagger\| \|F'(x_k)\| \|x_{k+1} - x_k\|. \end{aligned}$$

Due the fact that F is continuously differentiable, we can take limit in the last inequality to conclude that $F'(x_*)^\dagger F(x_*) = 0$.

Since $x_k \in K(t_k)$, for all $k = 0, 1, \dots$, the inequality in (12), follows by applying item **ii** of the Lemma 11 with $x = x_{k-1}$ and $t = t_{k-1}$ and using the definitions in (16) and (25).

To end the proof, combined the second inequality in (11) with the last part of the Corollary 8. \square

Therefore, it follows from Corollaries 8 and 12 that all statements in Theorem 3 are valid.

3 Special cases

In this section, we present some special cases of Theorem 3.

3.1 Convergence result for $F'(x_0)$ surjective

In this section we present a theorem under the hypothesis that $F'(x_0)$ is surjective. In this case, we can use a majorant condition, which gives the propriety that $\{x_k\}$ is invariant under the function $\bar{F} \rightarrow A^\dagger F$, where $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is any surjective linear operator.

Theorem 13. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $F : \Omega \rightarrow \mathbb{R}^m$ a continuously differentiable function. Take $x_0 \in \Omega$ such that $\beta := \|F'(x_0)^\dagger F(x_0)\| > 0$ and $F'(x_0)$ is surjective. Suppose that there exist $R > 0$ and a continuously differentiable function $f : [0, R) \rightarrow \mathbb{R}$ such that, $B(x_0, R) \subseteq \Omega$,*

$$\|F'(x_0)^\dagger(F'(y) - F'(x))\| \leq \bar{f}'(\|y - x\| + \|x - x_0\|) - \bar{f}'(\|x - x_0\|), \quad (28)$$

for any $x, y \in B(x_0, R)$, $\|x - x_0\| + \|y - x\| < R$,

h1) $\bar{f}(0) = 0$, $\bar{f}'(0) = -1$;

h2) \bar{f}' is convex and strictly increasing.

Consider the auxiliary function $h_\beta : [0, R) \rightarrow \mathbb{R}$,

$$h_\beta(t) := \beta + \bar{f}(t). \quad (29)$$

If h_β satisfies

h3) $h_\beta(t) = 0$ for some $t \in (0, R)$,

then $h_\beta(t)$ has a smallest zero $\bar{t}_* \in (0, R)$, the sequences for solving $h_\beta(t) = 0$ and $F(x) = 0$, with starting point $t_0 = 0$ and x_0 , respectively,

$$t_{k+1} = t_k - h'_\beta(t_k)^{-1} h_\beta(t_k), \quad x_{k+1} = x_k - F'(x_k)^\dagger F(x_k), \quad k = 0, 1, \dots, \quad (30)$$

are well defined, $\{t_k\}$ is strictly increasing, is contained in $[0, t_*)$, and converges Q -linearly to \bar{t}_* , $\{x_k\}$ is contained in $B(x_0, \bar{t}_*)$, and converges R -linearly to a point $x_* \in B[x_0, \bar{t}_*]$ such that $F'(x_*)^\dagger F(x_*) = 0$,

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \quad \|x_* - x_k\| \leq \bar{t}_* - t_k, \quad k = 0, 1, \dots, \quad (31)$$

$$\begin{aligned}\|x_{k+1} - x_k\| &\leq \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2} \|x_k - x_{k-1}\|^2, \quad k = 1, 2, \dots, \\ \|F'(x_0)^\dagger F(x_k)\| &\leq \left(\frac{t_{k+1} - t_k}{t_k - t_{k-1}} \right) \|F'(x_0)^\dagger F(x_{k-1})\|, \quad k = 1, 2, \dots.\end{aligned}\quad (32)$$

If, additionally, $h'_\beta(t_*) < 0$, then the sequences $\{t_k\}$ and $\{x_k\}$ converge Q -quadratically and R -quadratically to t_* and x_* , respectively.

Proof. Let $\bar{F} : \Omega \rightarrow \mathbb{R}^m$ be defined by

$$\bar{F}(x) = F'(x_0)^\dagger F(x), \quad x \in \Omega. \quad (33)$$

Under the hypothesis of the theorem, we will prove that \bar{F} satisfies all assumptions of the Theorem 3. Hence, with the exception of (32), the statements of the theorem follow from Theorem 3.

First of all, as $F'(x_0)$ is surjective, it follows from (5) that

$$F'(x_0)F'(x_0)^\dagger = I_{\mathbb{R}^m}, \quad (F'(x_0)F'(x_0)^\dagger)^\dagger = F'(x_0)F'(x_0)^\dagger. \quad (34)$$

Now, take $x \in B[x_0, t]$, $0 \leq t \leq \bar{t}_*$. Using the assumptions (28), **h1** and **h2**, we obtain

$$\|F'(x_0)^\dagger [F'(x) - F'(x_0)]\| \leq \bar{f}'(\|x - x_0\|) - \bar{f}'(0) \leq \bar{f}'(t) + 1 < 1.$$

Using Lemma 1, the above equation and the first equation in (34), we conclude that $(I_{\mathbb{R}^n} - F'(x_0)^\dagger(F'(x_0) - F'(x)))$ is non-singular and

$$\|(I_{\mathbb{R}^n} - F'(x_0)^\dagger(F'(x_0) - F'(x)))^{-1}\| \leq \frac{1}{1 - (\bar{f}'(t) + 1)} = -\frac{1}{\bar{f}'(t)}. \quad (35)$$

Again, the first equation in (34) implies that $F'(x) = F'(x_0)(I_{\mathbb{R}^n} - F'(x_0)^\dagger(F'(x_0) - F'(x)))$, which, using $F'(x_0)$ is surjective and $(I_{\mathbb{R}^n} - F'(x_0)^\dagger(F'(x_0) - F'(x)))$ is non-singular, yields $F'(x)$ is surjective for all $x \in B(x_0, \bar{t}_*)$. Hence, using (33) and properties of the Moore-Penrose inverse, we have

$$(\bar{F}'(x))^\dagger = (F'(x_0)^\dagger F'(x))^\dagger = F'(x)^\dagger F'(x_0), \quad \forall x \in \Omega.$$

The latter inequality implies that \bar{F}' satisfies (6) with $\kappa = 0$ and the second sequence in (30) coincides with the second sequence in (10). Moreover, using (33), (34) and (4), we obtain

$$\|\bar{F}'(x_0)^\dagger \bar{F}'(x_0)\| = \|(F'(x_0)^\dagger F'(x_0))^\dagger F'(x_0)^\dagger F'(x_0)\| = \|F'(x_0)^\dagger F'(x_0)\| \quad (36)$$

and

$$\|\bar{F}'(x_0)^\dagger\| = \|F'(x_0)^\dagger F'(x_0)\| = \|\Pi_{\text{Ker}(F'(x_0))^\perp}\| = 1. \quad (37)$$

Accordingly, (36) implies that $\|\bar{F}'(x_0)^\dagger \bar{F}'(x_0)\| > 0$, and (37) together with (28) and (33) implies that \bar{F}' satisfies (8) with $f = \bar{f}$.

Therefore, with the exception (32), the result of the theorem follow from Theorem 3 with $F = \bar{F}$, $f = \bar{f}$, $h_{\beta,\lambda} = h_\beta$, $\lambda = 0$ and $t_* = \bar{t}_*$.

Our task is now to show that (32) holds.

Take $k \in \{1, 2, \dots\}$. Using the first equation in (34), it follows by simple calculus that

$$F'(x_{k-1})^\dagger F'(x_0) (I_{\mathbb{R}^n} - F'(x_0)^\dagger (F'(x_0) - F'(x_{k-1}))) = F'(x_{k-1})^\dagger F'(x_{k-1}),$$

which, combined with (4), (35) and $\|x_{k-1} - x_0\| \leq t_{k-1} \leq \bar{t}_*$, yields

$$\begin{aligned} \|F'(x_{k-1})^\dagger F'(x_0)\| &\leq \|\Pi_{\text{Ker}(F'(x_{k-1}))^\perp} (I_{\mathbb{R}^n} - F'(x_0)^\dagger (F'(x_0) - F'(x_{k-1})))^{-1}\| \\ &\leq \|(I_{\mathbb{R}^n} - F'(x_0)^\dagger (F'(x_0) - F'(x_{k-1})))^{-1}\| \\ &\leq -(h'_\beta(t_{k-1}))^{-1}. \end{aligned}$$

Hence, using (30) and the first equation in (34), we obtain

$$\|x_k - x_{k-1}\| = \|F'(x_{k-1})^\dagger F(x_{k-1})\| \leq -(h'_\beta(t_{k-1}))^{-1} \|F'(x_0)^\dagger F(x_{k-1})\|. \quad (38)$$

Since $F(x_{k-1})$ is also surjective, it follows from (5) that $F'(x_{k-1})F'(x_{k-1})^\dagger = I_{\mathbb{R}^m}$, which combined with Lemma 10 and (31) gives

$$\begin{aligned} \|F'(x_0)^\dagger F(x_k)\| &= \|F'(x_0)^\dagger (F(x_k) - F(x_{k-1}) - F'(x_{k-1})(x_k - x_{k-1}))\| \\ &= \|F'(x_0)^\dagger \|E_F(x_{k-1}, x_k)\| \\ &\leq e_f(t_{k-1}, t_k) \frac{\|x_k - x_{k-1}\|}{(t_k - t_{k-1})} \\ &= h_\beta(t_k) \frac{\|x_k - x_{k-1}\|}{(t_k - t_{k-1})}, \end{aligned}$$

where the latter equation is obtained by combining (18), (29) and (30). Taking into account the last inequality, (38), $\{t_k\}$ and h'_β are strictly increasing, we have

$$\begin{aligned} \|F'(x_0)^\dagger F(x_k)\| &\leq -\frac{h_\beta(t_k)}{h'_\beta(t_{k-1})} \frac{\|F'(x_0)^\dagger F(x_{k-1})\|}{(t_k - t_{k-1})} \\ &\leq -\frac{h_\beta(t_k)}{h'_\beta(t_k)} \frac{\|F'(x_0)^\dagger F(x_{k-1})\|}{(t_k - t_{k-1})}. \end{aligned}$$

Therefore, the last inequality, together with the definition of $\{t_k\}$ in (30), imply the desired inequality. \square

3.2 Convergence result for Lipschitz condition

In this section, we first present a theorem corresponding to Theorem 3, but under the Lipschitz condition instead of the general assumption (8). We also present a theorem corresponding to Theorem (13), but under the Lipschitz condition instead of assumption (28).

Theorem 14. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $F : \Omega \rightarrow \mathbb{R}^m$ a continuously differentiable function. Suppose that*

$$\|F'(y)^\dagger(I_{\mathbb{R}^m} - F'(x)F'(x)^\dagger)F(x)\| \leq \kappa\|x - y\|, \quad \forall x, y \in \Omega$$

for some $0 \leq \kappa < 1$. Take $x_0 \in \Omega$ such that $\beta := \|F'(x_0)^\dagger F(x_0)\| > 0$, $F'(x_0) \neq 0$ and

$$\text{rank}(F'(x)) \leq \text{rank}(F'(x_0)), \quad \forall x \in \Omega.$$

Suppose that there exist $R > 0$ and $L > 0$, such that $B(x_0, R) \subseteq \Omega$,

$$\|F'(x_0)^\dagger\| \|F'(x) - F'(y)\| \leq L\|x - y\|, \quad \forall x, y \in B(x_0, R)$$

Take $\lambda = (1 - \beta L)\kappa$ and consider the auxiliary function $h_{\beta, \lambda} : [0, R) \rightarrow \mathbb{R}$,

$$h_{\beta, \lambda}(t) := \beta - (1 - \lambda)t + (Lt^2)/2.$$

If

$$\beta L \leq \Delta := \frac{(1 - \kappa)^2}{(\kappa^2 - \kappa + 1) + \sqrt{2\kappa^2 - 2\kappa + 1}},$$

then $h_{\beta, \lambda}(t)$ has a smallest zero $t_* = (1 - \lambda - \sqrt{(1 - \lambda)^2 - 2\beta L})/L$, the sequences for solving $h_{\beta, \lambda}(t) = 0$ and $F(x) = 0$, with starting point $t_0 = 0$ and x_0 , respectively,

$$t_{k+1} = t_k - h'_{\beta, 0}(t_k)^{-1} h_{\beta, \lambda}(t_k), \quad x_{k+1} = x_k - F'(x_k)^\dagger F(x_k), \quad k = 0, 1, \dots,$$

are well defined, $\{t_k\}$ is strictly increasing, is contained in $[0, t_*)$, and converges to t_* , $\{x_k\}$ is contained in $B(x_0, t_*)$, converges to a point $x_* \in B[x_0, t_*]$ such that $F'(x_*)^\dagger F(x_*) = 0$ and

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \quad \|x_* - x_k\| \leq t_* - t_k, \quad k = 0, 1, \dots,$$

$$\|x_{k+1} - x_k\| \leq \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2} \|x_k - x_{k-1}\|^2, \quad k = 1, 2, \dots$$

Moreover, if $\lambda = 0$ ($\lambda = 0$ and $h'_{\beta, 0}(t_*) < 0$), then the sequences $\{t_k\}$ and $\{x_k\}$ converge Q -linearly and R -linearly (Q -quadratically and R -quadratically) to t_* and x_* , respectively.

Proof. It is immediate to prove that F , x_0 and $f : [0, R) \rightarrow \mathbb{R}$ defined by $f(t) = Lt^2/2 - t$, satisfy the inequality (8), conditions **h1** and **h2**. Hence,

$$h_{\beta, \lambda}(t) := \beta - (1 - \lambda)t + (Lt^2)/2 = \beta + \lambda t + f(t).$$

Since,

$$\beta L \leq \Delta = \frac{(1 - \kappa)^2}{(\kappa^2 - \kappa + 1) + \sqrt{2\kappa^2 - 2\kappa + 1}} = \frac{(1 - \kappa)^2}{(1 - \kappa)^2 + \kappa + \sqrt{2\kappa^2 - 2\kappa + 1}} \leq 1, \quad (39)$$

we have $\lambda = (1 - \beta L)\kappa \geq 0$ and $\lambda = -\kappa f'(\beta)$. Moreover, the first inequality in (39) implies that $(1 - \lambda)^2 - 2\beta L \geq 0$, i.e., $h_{\beta, \lambda}$ satisfies **h3** and $t_* = (1 - \lambda - \sqrt{(1 - \lambda)^2 - 2\beta L})/L$ is its smallest root.

Therefore, taking f , $h_{\beta, \lambda}$, λ and t_* as defined above, all the statements of the theorem follow from Theorem 3. \square

Under the Lipschitz condition, Theorem 13 becomes:

Theorem 15. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $F : \Omega \rightarrow \mathbb{R}^m$ a continuously differentiable function. Take $x_0 \in \Omega$ such that $\beta := \|F'(x_0)^\dagger F(x_0)\| > 0$ and $F'(x_0)$ is surjective. Suppose that there exist $R > 0$ and $L > 0$, such that $B(x_0, R) \subseteq \Omega$,*

$$\|F'(x_0)^\dagger(F'(x) - F'(y))\| \leq L\|x - y\|, \quad \forall x, y \in B(x_0, R)$$

Consider the auxiliary function $h_\beta : [0, R) \rightarrow \mathbb{R}$,

$$h_\beta(t) := \beta - t + (Lt^2)/2.$$

If $\beta L \leq 1/2$, then $h_\beta(t)$ has a smallest zero $\bar{t}_ = (1 - \sqrt{1 - 2\beta L})/L$, the sequences for solving $h_\beta(t) = 0$ and $F(x) = 0$, with starting point $t_0 = 0$ and x_0 , respectively,*

$$t_{k+1} = t_k - h'_\beta(t_k)^{-1}h_\beta(t_k), \quad x_{k+1} = x_k - F'(x_k)^\dagger F(x_k), \quad k = 0, 1, \dots,$$

are well defined, $\{t_k\}$ is strictly increasing, is contained in $[0, t_)$, and converges Q -linearly to \bar{t}_* , $\{x_k\}$ is contained in $B(x_0, \bar{t}_*)$, and converges R -linearly to a point $x_* \in B[x_0, \bar{t}_*]$ such that $F'(x_*)^\dagger F(x_*) = 0$,*

$$\begin{aligned} \|x_{k+1} - x_k\| &\leq t_{k+1} - t_k, & \|x_* - x_k\| &\leq \bar{t}_* - t_k, & k = 0, 1, \dots, \\ \|x_{k+1} - x_k\| &\leq \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2} \|x_k - x_{k-1}\|^2, & k = 1, 2, \dots, \\ \|F'(x_0)^\dagger F(x_k)\| &\leq \left(\frac{t_{k+1} - t_k}{t_k - t_{k-1}} \right) \|F'(x_0)^\dagger F(x_{k-1})\|, & k = 1, 2, \dots. \end{aligned}$$

If, additionally, $\beta L < 1/2$, then the sequences $\{t_k\}$ and $\{x_k\}$ converge Q -quadratically and R -quadratically to t_ and x_* , respectively.*

Proof. The proof follows the same pattern as the proof of the Theorem 14. \square

3.3 Convergence result under Smale's condition

In this section, we first present a theorem corresponding to Theorem 3, but under Smale's α -condition, see [2, 3, 15]. We also present a theorem corresponding to Theorem (13), but under Smale's α -condition instead of the assumption (28).

To simplify, we take $\lambda = \kappa$ in the next theorem. As seen in Remark 1, this is always a possible choice for λ .

Theorem 16. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $F : \Omega \rightarrow \mathbb{R}^m$ an analytic function. Suppose that*

$$\|F'(y)^\dagger(I_{\mathbb{R}^m} - F'(x)F'(x)^\dagger)F(x)\| \leq \kappa\|x - y\|, \quad \forall x, y \in \Omega$$

for some $0 \leq \kappa < 1$. Take $x_0 \in \Omega$ such that $\beta := \|F'(x_0)^\dagger F(x_0)\| > 0$, $F'(x_0) \neq 0$ and

$$\text{rank}(F'(x)) \leq \text{rank}(F'(x_0)), \quad \forall x \in \Omega.$$

Suppose that

$$\gamma := \|F'(x_0)^\dagger\| \sup_{n \geq 1} \left\| \frac{F^{(n)}(x_0)}{n!} \right\|^{1/(n-1)} < +\infty, \quad B(x_0, 1/\gamma) \subseteq \Omega. \quad (40)$$

Consider the auxiliary function $h_{\beta, \kappa} : [0, 1/\gamma) \rightarrow \mathbb{R}$,

$$h_{\beta, \kappa}(t) := \beta - (2 - \kappa)t + t/(1 - \gamma t).$$

If

$$\alpha := \beta\gamma \leq 3 - 2\sqrt{2},$$

then $h_{\beta, \kappa}(t)$ has a smallest zero $t_ = (1 - \kappa + \alpha - \sqrt{(1 - \kappa + \alpha)^2 - 4(2 - \kappa)\alpha})/(2\gamma(2 - \kappa))$, the sequences for solving $h_{\beta, \kappa}(t) = 0$ and $F(x) = 0$, with starting point $t_0 = 0$ and x_0 , respectively,*

$$t_{k+1} = t_k - h'_{\beta, 0}(t_k)^{-1}h_{\beta, \kappa}(t_k), \quad x_{k+1} = x_k - F'(x_k)^\dagger F(x_k), \quad k = 0, 1, \dots,$$

are well defined, $\{t_k\}$ is strictly increasing, is contained in $[0, t_)$, and converges to t_* , $\{x_k\}$ is contained in $B(x_0, t_*)$, converges to a point $x_* \in B[x_0, t_*]$ such that $F'(x_*)^\dagger F(x_*) = 0$ and*

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \quad \|x_* - x_k\| \leq t_* - t_k, \quad k = 0, 1, \dots,$$

$$\|x_{k+1} - x_k\| \leq \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2} \|x_k - x_{k-1}\|^2, \quad k = 1, 2, \dots$$

Moreover, if $\kappa = 0$ ($\kappa = 0$ and $h'_{\beta, 0}(t_) < 0$), then the sequences $\{t_k\}$ and $\{x_k\}$ converge Q -linearly and R -linearly (Q -quadratically and R -quadratically) to t_* and x_* , respectively.*

We need the following results to prove the above theorem.

Lemma 17. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $F : \Omega \rightarrow \mathbb{R}^m$ an analytic function. Suppose that $x_0 \in \mathbb{R}^n$ and γ is defined in (41). Then, for all $x \in B(x_0, 1/\gamma)$ it holds that*

$$\|F'(x_0)^\dagger\| \|F''(x)\| \leq (2\gamma)/(1 - \gamma\|x - x_0\|)^3.$$

Proof. The proof follows the same pattern as the proof of Lemma 21 of [9]. \square

Lemma 18. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $F : \Omega \rightarrow \mathbb{R}^m$ be twice continuously differentiable on Ω . If there exists a $f : [0, R) \rightarrow \mathbb{R}$ twice continuously differentiable and satisfying*

$$\|F'(x_0)^\dagger\| \|F''(x)\| \leq f''(\|x - x_0\|),$$

for all $x \in \Omega$ such that $\|x - x_0\| < R$, then F and f satisfy (8).

Proof. The proof follows the same pattern as the proof of Lemma 22 of [9]. \square

Proof of Theorem 16. Consider the real function $f : [0, 1/\gamma) \rightarrow \mathbb{R}$ defined by

$$f(t) = \frac{t}{1 - \gamma t} - 2t.$$

It is straightforward to show that f is analytic and that

$$f(0) = 0, \quad f'(t) = 1/(1 - \gamma t)^2 - 2, \quad f'(0) = -1, \quad f''(t) = (2\gamma)/(1 - \gamma t)^3, \quad f^n(0) = n! \gamma^{n-1},$$

for $n \geq 2$. It follows from the latter equalities that f satisfies **h1** and **h2**. Moreover, as $f''(t) = (2\gamma)/(1 - \gamma t)^3$, combining Lemmas 17 and 18, we have F and f satisfy (8) with $R = 1/\gamma$. Hence,

$$h_{\beta, \lambda}(t) := \beta - (2 - \lambda)t + t/(1 - \gamma t) = \beta + \lambda t + f(t).$$

Since $\lambda = \kappa$, we have $0 \leq \lambda < 1$ and $\lambda = -\kappa f'(0) \geq -\kappa f'(\beta)$, where the latter inequality follows from **h2**. Moreover, $\alpha = \beta\gamma \leq 3 - 2\sqrt{2}$ implies that $((1 - \kappa + \alpha)^2 - 4(2 - \kappa)\alpha) \geq 0$, i.e., $h_{\beta, \lambda}$ satisfies **h3** and $t_* = (1 - \kappa + \alpha - \sqrt{(1 - \kappa + \alpha)^2 - 4(2 - \kappa)\alpha})/(2\gamma(2 - \kappa))$ is its smallest root.

Therefore, taking f , $h_{\beta, \lambda}$, λ and t_* as defined above, all the statements of the theorem follow from Theorem 3. \square

Under the Smale's α -condition, Theorem 13 becomes:

Theorem 19. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $F : \Omega \rightarrow \mathbb{R}^m$ an analytic function. Take $x_0 \in \Omega$ such that $\beta := \|F'(x_0)^\dagger F(x_0)\| > 0$ and $F'(x_0)$ is surjective. Suppose that

$$\gamma := \sup_{n \geq 1} \left\| \frac{F'(x_0)^\dagger F^{(n)}(x_0)}{n!} \right\|^{1/(n-1)} < +\infty, \quad B(x_0, 1/\gamma) \subseteq \Omega. \quad (41)$$

Consider the auxiliary function $h_{\beta, \kappa} : [0, 1/\gamma) \rightarrow \mathbb{R}$,

$$h_{\beta, \kappa}(t) := \beta - 2t + t/(1 - \gamma t).$$

If

$$\alpha := \beta\gamma \leq 3 - 2\sqrt{2},$$

then $h_{\beta, \kappa}(t)$ has a smallest zero $t_* = (1 + \alpha - \sqrt{(1 + \alpha)^2 - 8\alpha})/(4\gamma)$, the sequences for solving $h_{\beta, \kappa}(t) = 0$ and $F(x) = 0$, with starting point $t_0 = 0$ and x_0 , respectively,

$$t_{k+1} = t_k - h'_{\beta, 0}(t_k)^{-1} h_{\beta, \kappa}(t_k), \quad x_{k+1} = x_k - F'(x_k)^\dagger F(x_k), \quad k = 0, 1, \dots,$$

are well defined, $\{t_k\}$ is strictly increasing, is contained in $[0, t_*)$, and converges Q -linearly to \bar{t}_* , $\{x_k\}$ is contained in $B(x_0, \bar{t}_*)$ and converges R -linearly to a point $x_* \in B[x_0, \bar{t}_*]$ such that $F'(x_*)^\dagger F(x_*) = 0$,

$$\|x_{k+1} - x_k\| \leq t_{k+1} - t_k, \quad \|x_* - x_k\| \leq \bar{t}_* - t_k, \quad k = 0, 1, \dots,$$

$$\|x_{k+1} - x_k\| \leq \frac{t_{k+1} - t_k}{(t_k - t_{k-1})^2} \|x_k - x_{k-1}\|^2, \quad k = 1, 2, \dots,$$

$$\|F'(x_0)^\dagger F(x_k)\| \leq \left(\frac{t_{k+1} - t_k}{t_k - t_{k-1}} \right) \|F'(x_0)^\dagger F(x_{k-1})\|, \quad k = 1, 2, \dots$$

If, additionally, $\alpha := \beta\gamma < 3 - 2\sqrt{2}$, then the sequences $\{t_k\}$ and $\{x_k\}$ converge Q -quadratically and R -quadratically to t_* and x_* , respectively.

Proof. The proof follows the same pattern as the proof of Theorem 16. \square

4 Final remarks

We presented a new semi-local convergence analysis of the Gauss-Newton method for solving (1), where F satisfies (3), under a majorant condition. It would also be interesting to present a local convergence analysis of the Gauss-Newton method, under a majorant condition, for the problem under consideration. As a consequence, we would get convergence results for analytical functions under an γ -condition. This local analysis will be performed in the future.

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